Links between al-Khwārizmī's method of multiplying and dividing fractions and Brahmagupta's arithmetic

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Abstract: *Muḥammad ibn Mūsā al-Khwārizmī's treatise on arithmetic is the oldest known Arabic text on the Indian decimal place value system. Written around c. 825 CE after his treatise on algebra, it survives only in Latin translations. Known as Dixit Algorizmi or DA (Al-Khwarizmi said), the Latin manuscript was incomplete. It abruptly ends partway through the 12th of 19 chapters during an explanation of* $3\frac{1}{2} \times 8\frac{3}{11}$. *Presumably long lost, there was no English translation of either the multiplication or division of mixed fractions. Yet, a complete manuscript of Dixit Algorizmi was subsequently found in the museum of the Hispanic Society of America in New York. Translated and published in German in 1997, al-Khwārizmī's method of multiplying and dividing mixed fractions is now presented in English.*

Summary

The manuscript of al-Khwārizmī's [c. 780 – c. 850 CE] 'arithmetic' previously translated into English is held in the Cambridge University Library with the first English translation appearing in 1990 [1]. Being incomplete, English translations of al-Khwārizmī's method of multiplying and dividing mixed fractions are thought to have been unavailable until now. While multiplying fractions is a simple matter, a rule often taught when dividing fractions is *invert and multiply*. Was this al-Khwārizmī's method? Notably, we see al-Khwārizmī did not depict a division procedure in which he inverted and multiplied the divisor, while earlier Indian writers such as Brahmagupta [c. 598 – c. 668 CE] did. Instead, as with fraction multiplication, al-Khwārizmī converted both terms into improper fractions with the same denominator and divided the new numerators to arrive at the answer. Here now follow English translations of al-Khwārizmī's given examples.

Al- Khwārizmī on the multiplication of mixed fractions. $3\frac{1}{2} \times 8\frac{3}{11}$

<12> **Capitulum aliud in multiplicatone fractionum et divisione earum.** [2]	<12> **Another chapter on the multiplication of fractions and their division.** [3]
⟨12.3⟩ Cum ergo volueris multiplicare tres et dimidium in VIII et tribus partibus de XI, scribe tres. Postea pone sub eis unum et sub uno duo, et cum hoc feceris, iam posuisti tres et dimidium, quia dimidium est una pars ex duabus, quemadmodum unum minutum est sexagesima unius. Post hec scribe in alia parte VIII et sub eis III et sub tribus XI, et cum hoc feceris, iam posuisti VIII et tres partes de XI. 3 8 1 3 2 11	<12.3> So if you want to multiply three and {a one} half by eight and three {eleventh parts}, write 'three'. Then put under it (lit. them) a 'one', and under the 'one' 'two', and when you have done this you have written down 'three and {a one} half', since a half is one part out of two, just as a minute is a sixtieth of one. Next write on the other side 'eight' and under that (lit. them) 'three' and under the 'three' 'eleven', and when you have done that you have written down 'eight and three elevenths'. 3 8 1 3 2 11

Et sic reddas unumquemque ex eis de genere ultime differentie, hoc est, multiplicabis tria in duobus, ad que refertur unum, augesque eis unum, et sic erunt medietates.	And thus rewrite each of them according to the type in the bottom position, that is, you multiply the 'three' by the 'two' (to which the 'one' refers) and then add one: these will be halves.
Erunt igitur VII medietates, quas scribes in loco trium et destrues tria atque unum quod est sub eis.	So there are seven halves which you write in place of the 'three' and cross out the 'three' and the 'one' which is beneath it.
Multiplicabis VIII quoque in undecim, ad que referentur tria, et adde super ea tria, et sic reddes ea partes de XI.	Also, you multiply the 'eight' by the 'eleven' (to which the 'three' refers) and then add three and in this way you make them elevenths.
Eruntque LX⟨L⟩I, que scribes in loco VIII et destrues VIII atque tria que sunt sub eis.	There are ninety-one [elevenths] which you write in place of the 'eight', crossing out the 'eight' and the 'three'.
Deinde multiplicabis genus medietatum, que sunt duo, in genere partium, que sunt XI, et erunt X⟨X⟩II.	Next, you multiply the type of 'half', which is 2, by the type of 'eleventh', which is 11, to get 22.
Eritque hoc ex genere secundorum, et servabis illud.	This is the type of the parts (of unity) which is retained.
Item multiplicabis VII medietates in LX⟨L⟩ una parte XI, et erunt DCXXXVII.	Next, you multiply 7 (halves) by 91 (elevenths) getting 637.
Et sunt etiam ex genere secundorum, que divides per XXII, et erunt eiusdem generis.	And these are the type of parts of unity which are 'twenty-secondths', and they are (all) of the same type.
Divide igitur unum ex eis super alium, et quod exierit erit numerus integer, et quod remanserit erunt partes unius de illo numero quem dividis.	Thus divide one of them by the other and what results will be a whole number, with a remainder of parts of unity of the type of the dividing number.
Et hoc est quod exivit tibi de multiplicatione, XXVIII scilicet et XX una pars ex XX duabus partibus unius.	And this is what you got out of the multiplication, 28 and 21 twenty-secondth parts of unity.
Et similiter erit universa multiplicatio fractionum.	And all multiplication of fractions goes similarly.

STEP-BY-STEP MULTIPLICATION EXPLANATION

To multiply $3\frac{1}{2} \times 8\frac{3}{11}$, write down the following:

3	8
1	3
2	11

From the depiction of 3 and $\frac{1}{2}$ in the left column, we calculate the number of ‘halves’ as (3 × 2) + 1 = 7. From the depiction of 8 and $\frac{3}{11}$ in the right column, we calculate the number of ‘elevenths’ as (8 × 11) + 3 = 91.

The 3 is thus covered in ‘dust’ and replaced by 7 and the 8 is covered in dust and replaced by 91. The 1 and 3 are presumably both erased and covered in dust as they now serve no purpose.

7 91

2 11

Next, multiply the two numbers in the bottom line, giving 22, which is the 'type' of the parts of unity we now have, namely, 'twenty-secondths'. Then, multiply the two numbers in the top line, getting 637. This is the number of '22ndths' we have.

637

22

Next, 637 divided by 22 yields 28 wholes (units, ones) with 21 remaining, which are '22ndths'. So, from the method of al-Khwarizmi, the result of $3\frac{1}{2} \times 8\frac{3}{11}$ is $28\frac{21}{22}$.

Al-Khwārizmī's Indian Source

The original Latinised name of al-Khwārizmī's text was *Algorithmo de Numero Indorum,* reflecting its Indian roots. By way of example, we can see Brahmagupta's explanation from his 628 CE *Brāhmasphuṭasiddhānta* in al-Khwārizmī's example.

From Brahmagupta, [4] we read, *"Integers are multiplied by the denominators and have the numerators added."* From this, in $3\frac{1}{2}$ al-Khwārizmī multiplies 3×2 and adds 1 to arrive at

7

2

Then, for $8\frac{3}{11}$ al-Khwārizmī multiplies 8×11 and adds 3 to arrive at

7 91

2 11

Brahmagupta continues, "The product of the numerators, divided by the product of the denominators, is multiplication of two or of many terms." From this, al-Khwārizmī multiplies 7×91 for a numerator of 637 and 2×11 for a denominator of 22 to arrive at the fraction $\frac{637}{22}$ to arrive at $28\frac{21}{22}$.

Al- Khwārizmī on the division of mixed fractions. $20\frac{2}{13} \div 3\frac{1}{3}$

For brevity, the English translation is provided below with the Latin as an **Appendix**.

12.4
And if you want to divide in this way (i.e. with fractions), one <number> above another, place each <number> in the following way. After <you have done> this, divide one (i.e. the top one) by the other. And what remains will be a whole number [5].

For example: if you want to divide 20 and two parts of 13 i.e. $20\frac{2}{13}$ by 3 and one third i.e. $3\frac{1}{3}$, you would write the following:

3	20
1	2
3	13

And now we see that $\frac{1}{3}$ is different from $\frac{2}{13}$ since 13 does not have one third. Therefore, multiply the genus (denominator) of thirds with the other part, (which is 13), until they are of the same genus. And this genus will be 39.

Then multiply $20\frac{2}{13}$ with 39. And 20 multiplied with 39 is 780. And also add about that $\frac{2}{13}$ of 39, which is 6, because one-thirteenth of 39 is 3. And the result is 786.

And when you have done this, now you have the $20\frac{2}{13}$ th part of the genus 39. After this, also multiply $3\frac{1}{3}$ with the parts which are 39, until both are of the same genus. And so, 3 multiplied by 39 will be 117. To this add one third of 39 and the result is 130 parts of 39.

And now they have been made equal in the genus. Therefore, divide one above the other, just like 780 over 130, and what remains will be a whole number. In truth what will have been left over will be part of the same number through which you are dividing. And it will be the (whole) number 6 and 6 parts of 130.

If the numbers, which you want to divide, one through the other, are equal parts, like a quarter by a quarter or an eighth by an eighth or a seventeenth by a seventeenth, put them all back into the genus of that fraction because they are equal. After this divide that, which you want to divide about the other, and the result of this ought to be the number one. Do this in the same way in all situations, where you want to divide fractions or whole numbers and you will find <the answer> if God is willing.

STEP-BY-STEP DIVISION EXPLANATION

In al-Khwārizmī's example $20\frac{2}{13} \div 3\frac{1}{3}$ the mixed fractions are converted into the same denominations or part sizes, which are 39ths.

Thus, for the dividend, $20 = \frac{260}{13}$ which equals $\frac{780}{39}$ and $\frac{2}{13} = \frac{6}{39}$. So, altogether the dividend is $\frac{786}{39}$.

For the divisor, $3 = \frac{9}{3}$ which equals $\frac{117}{39}$ and $\frac{1}{3} = \frac{13}{39}$. So, altogether the divisor is $\frac{130}{39}$.

The problem $20\frac{2}{13} \div 3\frac{1}{3}$ thus becomes $\frac{786}{39} \div \frac{130}{39}$.

Al-Khwārizmī says because the genus (denominators) are now the same (i.e. 39^{ths}) the problem then becomes 786 ÷ 130 which in turn equals $6\frac{6}{130}$.

DISCUSSION

Today, the multiplication of two fractions does not represent a pedagogical problem. The two numerators are multiplied as are the two denominators. So, for example, $\frac{a}{b} \times \frac{c}{d}$ simply becomes

$\frac{ac}{bd}$. By applying the distributive law to al-Khwārizmī's example of $\mathbf{3\frac{1}{2} \times 8\frac{3}{11}}$ we can restate this as
$(3 + \frac{1}{2}) \times (8 + \frac{3}{11})$ which becomes
$(3 \times 8) + (3 \times \frac{3}{11}) + (\frac{1}{2} \times 8) + (\frac{1}{2} \times \frac{3}{11})$ which becomes
$24 + \frac{9}{11} + 4 + \frac{3}{22}$ which becomes
$28\frac{21}{22}$.

However, division is more problematic so students are often taught to convert division by a fraction into multiplication by a reciprocal. For example, $\frac{a}{b} \div \frac{c}{d}$ would become $\frac{a}{b} \times \frac{d}{c}$. For example, in the division $12 \div \frac{2}{3}$ the expression would change to become $12 \times \frac{3}{2}$. Notably, al-Khwārizmī's example is $20\frac{2}{13} \div 3\frac{1}{3}$ which initially becomes $\frac{262}{13} \div \frac{10}{3}$.

By applying the mnemonic *"Ours is not to reason why, just invert and multiply"* this would then become $\frac{263}{13} \times \frac{3}{10}$. This takes us straight to $\frac{786}{130}$ which in turn leads us to $6\frac{6}{130}$ which can be further simplified to $6\frac{3}{65}$ which al-Khwārizmī does not do.

Notably, Brahmagupta does refer to an inversion rule in his *Brāhmasphuṭasiddhānta*. On division, he wrote, *"Both terms being rendered homogeneous the denominator and numerator of the divisor are transposed: and then the denominator of the dividend is multiplied by the [new] denominator; and its numerator, by the [new] numerator. Thus division [is performed.]"* Elsewhere, we read, "The denominator and numerator of the divisor having been interchanged, the denominator of the dividend is multiplied by the (new) numerator. Thus division of proper fractions is performed" [6].

An example is given by the translator (Henry Colebrooke), where an area of a rectangle is $122\frac{1}{2}$ with a side of $10\frac{1}{2}$; tell the upright. Reduced to a homogeneous form, the terms become $\frac{245}{2}$ and $\frac{21}{2}$ with the former being the area and the latter a side length. In the division of the area by the side, the latter is transposed and the two terms $\frac{245}{2}$ and $\frac{2}{21}$ are then multiplied to give $\frac{490}{42}$, or $11\frac{2}{3}$ as the upright. (Note: The translator incorrectly gives an answer of $11\frac{1}{2}$.)

CONCLUSION

By exploring the methods al-Khwārizmī follows to multiply and divide mixed fractions, we can more easily see how some but not all ideas evolved over time in the Arabic world from ancient Indian roots.

References

[1] *Thus Spake al-Khwarizmi: A Translation of the Text of Cambridge University Library Ms. Ii.vi.5*, John N. Crossley and Alan S. Henry, Historia Mathematica, 17 (1990), 103-131.

[2] *Die älteste lateinische Schrift über das indische Rechnen nach al-Ḫwārizmī, (Latin & German)* Menso Folkerts; Paul Kunitzsch; Hispanic Society of America, München : Verlag der Bayerischen Akademie der Wissenschaften, 1997.

[3] Latin multiplication translation courtesy of Peter Crabtree, Melbourne Australia.

[4] *Algebra, with arithmetic and mensuration, from the Sanscrit of Brahmegupta and Bháscara, translated by Henry Thomas Colebrooke.* Brahmagupta, Bhāskarācārya, and Henry Thomas Colebrooke, London: J. Murray, 1817.

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[5] Latin division translation courtesy of Tess Anderson, Melbourne Australia.

[6] *Brāhma-sphuṭa-siddhānta / ed. by a board of editors headed by Acharyavara Ram Swarup Sharma.* Brahmagupta, Ram Swarup Sharma, Pṛthūdakasvāmin, and Sudhākara Dvivedī, Indian Institute of Astronomical and Sanskrit Research, New Delhi, 1966.

Appendix

Section 12.4 of the translation of Al-Khwarizmi's Arabic text on division of mixed numbers into Latin.

⟨12.4⟩ Et cum volueris dividere de hoc genere, id est de fractionibus, aliquid super aliud, pone utrumque ex uno genere. Post hec divide unum super aliud, et quod exierit erit numerus integer. Verbi gratia. Si velles dividere XX et duas partes de XIII super tria et terciam unius, que sic describeres:

3	20
1	2
3	13

Et iam notavimus quod tercia sit diversa a duabus partibus de XIII, non enim habent terciam. Multiplica ergo genus tercianorum, quod est tria, in differentia parcium que sunt de XIII, donec sint unius generis. Eritque genus XXXIX. Deinde multiplica XX et duas partes de XIII in XXXIX. Eruntque XX in XXXIX DCC octoginta. Et adde desuper duas partes XIII de XXXIX, que sunt VI, quia XIII pars de XXXIX sunt tria. Eruntque DCCLXXXVI. Cumque hoc feceris, iam reddidisti ipsa XX et duas partes de XIII ex genere XXXIX. Post hec multiplica etiam tria et terciam unius in partibus que sunt XXXIX, donec sint utrique unius generis. Erunt igitur tria in XXXIX CXVII, super que addes terciam partem XXXIX, que est XIII, et erunt centum XXX partes de XXXIX. Iamque equaverunt in equali genere. Divide ergo unum super aliud, scilicet septingenta LXXXVI super CXXX, et quod exierit erit numerus integer. Quod vero remanserit, erit pars eiusdem numeri, super quem dividis. Eruntque ex numero VI et VI partes ex centum XXX. Quod si numeri, quos vis dividere unum per alium, fuerint partes equales, ut quarte per quartas vel octave per octavas aut partes de XVII in partes de XVII, redde unumquemque in genus illius fractionis, quia sunt equales. Post hec divide ipsum quem vis dividere super alium, et quod exierit, illud est quod debetur uni. Similiter fac in universis, que volueris dividere ex fractionibus sive ex integro numero, et invenies, si deus voluerit.